Convex Optimization

(EE227A: UC Berkeley)

Lecture 26 Interior point methods

25 Apr, 2013

Suvrit Sra

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- Interior Point Methods build on the Newton method to ultimately tackle the above convex optimization problem

Preliminaries

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- ▶ What is complexity of such a scheme?
- Numerical problems when t_k becomes large; breakdown?
- ► Standard theory of unconstrained minimization predicts slowdown as t_k becomes larger ...

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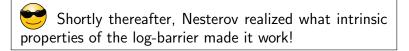
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Lemma Let $\{x_k\}$ be generated by Newton method for f: $x_{k+1} = x_k - [f''(x_k)]^{-1}f'(x_k) \quad k \ge 0.$ Let $\{y_k\}$ be seq. generated by NM for ϕ : $y_{k+1} = y_k - [\phi''(y_k)]^{-1}\phi'(y_k),$ with $Ay_0 = x_0$. Then, $Ay_k = x_k$ for all $k \ge 0.$

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Newton method remains same—strong contrast to gradient method! **Stopping condition:**

$$\langle [f''(x_k)]^{-1} f'(x_k), f'(x_k) \rangle < \epsilon$$

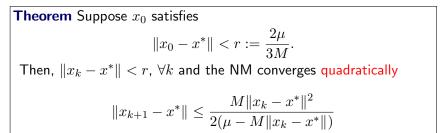
independent of choice of basis A!

Assumptions

- Lipschitz Hessian: $\|\nabla^2 f(x) \nabla^2 f(y)\| \le M \|x y\|$
- Local strong convexity: There exists a local minimum x^* with

$$\nabla^2 f(x^*) \succeq \mu I, \qquad \mu > 0.$$

• Locality: Starting point x_0 "close enough" to x^*



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- \blacktriangleright Convergence analysis depends on $\mu,$ and M
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- Mismatch between geometry of method and its convergence analysis

What's missing

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 ${}^{\mathbf{k} \mathbf{\widetilde{r}}}$ Thus, at $x \in \operatorname{dom} f$, and any $u, v \in \mathbb{R}^n$ we have

 $\langle f^{\prime\prime\prime}(x)[u]v, v\rangle \le M \|u\| \|v\|^2$

$$\begin{array}{l} \textcircled{\mbox{W}} \\ \blacksquare \end{array} \ {\rm Using} \ x \leftarrow Ay, \ u' \leftarrow Au, \ v' \leftarrow Av, \ \phi(y) = f(Ay) \\ \\ \langle f'''(x)[u]v, \ v \rangle = \langle \phi'''(x)[u']v', \ v' \rangle \end{array}$$

$$\begin{array}{l} \textcircled{\mbox{Solution}} & \fbox{\mbox{Using } x \leftarrow Ay, \ u' \leftarrow Au, \ v' \leftarrow Av, \ \phi(y) = f(Ay) \\ & \langle f'''(x)[u]v, \ v \rangle = \langle \phi'''(x)[u']v', \ v' \rangle \end{array} \end{array}$$

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 $^{\tiny \hbox{\tiny W}}$ This brings us to the idea of self-concordance

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Derivatives

$$Df(x)[u] = \phi'(x;t) = \langle f'(x), u \rangle$$
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Note: Third derivative: symmetric trilinear operator, so it operates on $[u_1, u_2, u_3]$ to yield a trilinear symmetric form.

$$D^{p}f(x)[u_{1},\ldots,u_{p}] = \left.\frac{\partial^{p}}{\partial t_{1}\cdots\partial t_{p}}\right|_{t_{1}=\cdots=t_{p}=0} f(x+t_{1}u_{1}+\cdots+t_{p}u_{p})$$

Self-concordant functions and barriers

Def. (Self-concordant). Let \mathcal{X} be a closed convex set. A function $f : \operatorname{int}(\mathcal{X}) \to \mathbb{R}$ called self-concordant (SC) on \mathcal{X} if $f \in C^3(\mathcal{X})$ with $f(x_k) \to +\infty$ if $x_k \to \overline{x} \in \partial \mathcal{X}$ f satisfies the SC inequality $|D^3f(x)[u, u, u]| \le 2 \left(D^2f(x)[u, u]\right)^{3/2}, \quad \forall x \in \operatorname{int}(\mathcal{X}), u \in \mathbb{R}^n$

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Def. Given a real $\vartheta \ge 1$, F is called a ϑ -self-concordant barrier (SCB) for \mathcal{X} if F is SC and

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- ► Exponents 3/2 and 1/2 crucial—ensure both sides have same degree of homogeneity in *u* (for affine invariance)
- ► Factor 2 can be scaled by scaling f; chosen for convenience; equiv. to D²f Lipschitz with constant 2 in norm ||·||_{f''(x)}

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Example $f(x) = -\log x : \mathbb{R}_{++} \to \mathbb{R}$ is a 1-SCB for \mathbb{R}_+

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- Convex quadratic functions; f'''(x) = 0
- ► Log-barrier for $\phi(x) = a + \langle b, x \rangle \frac{1}{2}x^T Ax$; $f(x) = -\log \phi(x)$ Show: $|D^3 f(x)[u, u, u]| = |2\omega_1^3 + 3\omega_1\omega_2|$, where $\omega_1 = Df(x)[u]$, $\omega_2 = \frac{1}{\phi(x)}u^T Au$; also show that $D^2 f(x)[u, u] = \omega_1^2 + \omega_2$.

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Lemma A function f is SC iff for any $x \in int(\mathcal{X})$, and $u_1, u_2, u_3 \in \mathbb{R}^n$

 $|D^{3}f(x)[u_{1}, u_{2}, u_{3}]| \leq 2||u_{1}||_{f''(x)}||u_{1}||_{f''(x)}||u_{1}||_{f''(x)}$

Proof: Essentially generalized Cauchy-Schwarz (some work).

SC Optimization

Key quantities

- ▶ Let f(x) be SC, and that $f''(x) \succ 0$ within dom f
- \blacktriangleright Simplified notation for the local norms at x

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▶ Let us use these to state three crucial observations

Three key facts

At any point $x \in \text{dom} f = \text{int}(\mathcal{X})$, there is an ellipsoid

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 \square Within this ellipsoid (aka Dinkin ellipsoid), f is almost quadratic

$$r := \|u\|_x < 1 \implies (1-r)^2 f''(x) \preceq f''(x+u) \preceq \frac{1}{(1-r)^2} f''(x)$$

Three key facts

At any point $x \in \text{dom} f = \text{int}(\mathcal{X})$, there is an ellipsoid

$$W(x) := \{ y \in \mathbb{R}^n \mid ||y - x||_x \le 1 \} \subset \operatorname{dom} f.$$

Within this ellipsoid (aka Dinkin ellipsoid), f is almost quadratic

$$r := \|u\|_x < 1 \implies$$
$$(1-r)^2 f''(x) \preceq f''(x+u) \preceq \frac{1}{(1-r)^2} f''(x)$$

Moreover, it also holds that

 $f(x) + \langle f'(x), u \rangle + \rho(-r) \leq f(x+u) \leq f(x) + \langle f'(x), u \rangle + \rho(r),$ where $\rho(r) := -\log(1-r) - s = s^2/2 + s^3/3 + \cdots$ **Proof:** See Chap. 4 of Nesterov (2004).

Setting up Newton's method

Newton decrement

$$\lambda_f(x) := \langle [f''(x)]^{-1} f'(x), f'(x) \rangle^{1/2}.$$

Observe: $\lambda_f(x) = \|f'(x)\|_x^*$ (local, dual-norm of gradient).

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$$\lambda_f(x) = \max_u \left\{ Df(x)[u] \mid D^2 f(x)[u, u] \le 1 \right\}$$

- $\lambda_f(x)$ if a finite continuous function of $x \in \operatorname{dom} f$
- ▶ It vanishes at the (unique, if any) minimizer x_f^* of f on $\operatorname{dom} f$

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Theorem If $\lambda_f(x) < 1$ for some $x \in \text{dom } f$. Then, $\min f(x)$ s.t., $x \in \text{dom } f$, has a unique optimal solution.

1 Select $x_0 \in \operatorname{dom} f$

2 For
$$k \ge 0$$
: $x_{k+1} = x_k - \frac{1}{1+\lambda_f(x_k)} [f''(x_k)]^{-1} f'(x_k)$

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Theorem For any $k \ge 0$, the iterates of the damped NM satisfy $f(x_{k+1}) \le f(x_k) - \rho(-\lambda_f(x_k))$

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At each step, f(x) decreases by at least $\omega(\lambda)$

Damped Newton method

• Globally convegent; iteration complexity can be derived.

• Local quadratic convergence: $\lambda_f(x_{k+1}) \leq 2\lambda_f(x_k)^2$ for small enough $\lambda_f(x_k)$

 \bullet Though, better to start with DN, and switch to pure Newton after N iterations, where

$$N \le \frac{1}{\omega(\beta)[f(x_0) - f(x_f^*)]},$$

and $\lambda_f(x_k) \geq \beta$, where $\beta \in (0, 0.3819...)$

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Standard convex problem

min $c^T x \quad x \in \mathcal{X},$

where \mathcal{X} is a compact set for which dom $F \equiv \mathcal{X}$.

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$$x^*(t) = \underset{x \in \operatorname{dom} F}{\operatorname{argmin}} \quad tc^T x + F(x), \quad t \ge 0.$$

• Any point of the central path (set $\{x^*(t)\}$) satisfies

$$tc + F'(x^*(t)) = 0.$$

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▶ Aim is to iteratively find points close to central path

Minimization using SCBs

Approximate solution:

$$\lambda_{F_t}(x) := \|F'_t(x)\|_x^* = \|tc + F'(x)\|_x^* \le \beta,$$

where β is the centering parameter (approx. solution quality).

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where β is the centering parameter (approx. solution quality).

Theorem For any t > 0, we have $c^T x^*(t) - c^T x^* \le \frac{\vartheta}{t}$. If a point x is an approximate solution (close to $x^*(t)$), then $c^T x - c^T x^* \le \frac{1}{t} \left(\vartheta + \frac{\beta(\beta + \sqrt{\vartheta})}{1 - \beta} \right)$.

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2 At k-th iteration, set

$$t_{k+1} = t_k + \frac{\gamma}{\|c\|_{x_k}^*}, \quad \gamma = \frac{\sqrt{\beta}}{1 - \sqrt{\beta}} - \beta,$$

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Theorem Above scheme yields $c^T x_N - c^T x^* \leq \epsilon$ after no more than N steps, where

$$N \le O\left(\sqrt{\vartheta}\log \frac{\vartheta \|c\|_{x^*}^*}{\epsilon}
ight).$$

More

We've barely scratched the surface!

More

We've barely scratched the surface!

► Much more to interior point methods.

► See references for fuller picture.

Also read: Ch. 9,10,11 of BV for high-level overview.

References

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