Convex Optimization

(EE227A: UC Berkeley)

Lecture 20
(Coordinate descent)

04 Apr, 2013

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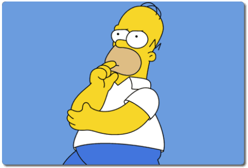
Admin

- ♡ HW3 **due right now!**
- ♡ HW4 is out! Please ask your Qs on Piazza
- ♡ Project 4 page reports due on **4/11/2013**
- ♡ Poster presentations: 3hrs: **When in May?**

Challenge problem

$$I(p) := \sqrt{p} \int_0^\infty \left| \frac{\sin x}{x} \right|^p dx$$



$$\boxed{\textbf{Minimize } I(p) \textbf{ over } p \geq 1}$$

Coordinate descent

$$\boxed{\text{So far: } \min f(x) = \sum_i f_i(x)}$$

Coordinate descent

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Since $x \in \mathbb{R}^n$, now consider

$$\min f(x) = f(x_1, x_2, \ldots, x_n)$$

Previously, we went through $f_1, \ldots, f_m$

What if we now go through $x_1, \ldots, x_n$ one by one?

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  - Pick an index i from $\{1, \ldots, n\}$

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- For $k = 0, 1, \ldots$
  - Pick an index i from $\{1, \ldots, n\}$
  - Optimize the ith coordinate

$$x_i^{k+1} \leftarrow \operatorname*{argmin}_{\xi \in \mathbb{R}} f(\underbrace{x_1^{k+1}, \ldots, x_{i-1}^{k+1}}_{\textbf{done}}, \underbrace{\xi}_{\textbf{current}}, \underbrace{x_{i+1}^k, \ldots, x_n^k}_{\textbf{todo}})$$

- Decide when/how to stop; *return* x^k

! x_i^{k+1} **overwrites** value in x_i^k (in actual implementation)

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- ♣ These days renewed interest in CD for large-scale problems
- ♣ Notice: in general CD is “derivative free”

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- ♣ **Random sampling**: pick random index at each iteration

Coordinate descent – Example

$$\min \|Ax - b\|_2^2$$

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Coordinate descent update

$$x_j \leftarrow \frac{\sum_{i=1}^m a_{ij}\left(b_i - \sum_{l \neq j} a_{il} x_l\right)}{\sum_{i=1}^m a_{ij}^2}$$

(dropped superscripts, since we overwrite)

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- ♠ Nonsmooth case more tricky

Block coordinate descent (BCD)

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) := f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \\ & \boldsymbol{x} \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_m. \end{aligned}$$

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Gauss-Seidel style

$$\boxed{\boldsymbol{x}_i^{k+1} \leftarrow \operatorname*{argmin}_{\boldsymbol{\xi} \in \mathcal{X}_i} f(\underbrace{\boldsymbol{x}_1^{k+1}, \ldots, \boldsymbol{x}_{i-1}^{k+1}}_{\textbf{done}}, \underbrace{\boldsymbol{\xi}}_{\textbf{current}}, \underbrace{\boldsymbol{x}_{i+1}^{k}, \ldots, \boldsymbol{x}_m^{k}}_{\textbf{todo}})}$$

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Jacobi style (easy to parallelize)

$$\boxed{\boldsymbol{x}_i^{k+1} \leftarrow \operatorname*{argmin}_{\boldsymbol{\xi} \in \mathcal{X}_i} f(\underbrace{\boldsymbol{x}_1^{k}, \ldots, \boldsymbol{x}_{i-1}^{k}}_{\textbf{don't clobber}}, \underbrace{\boldsymbol{\xi}}_{\textbf{current}}, \underbrace{\boldsymbol{x}_{i+1}^{k}, \ldots, \boldsymbol{x}_m^{k}}_{\textbf{todo}})}$$

BCD – convergence

Theorem Let f be continuously differentiable over $\mathcal{X} := \bigtimes_{i=1}^{m} \mathcal{X}_i$. Further, assume for each block i and $x \in \mathcal{X}$, the minimum

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is **uniquely attained**. Every limit point of the sequence $\{\boldsymbol{x}^k\}$ generated by BCD, is a stationary point of f.

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- **Unique solutions** of subproblems not always possible
- Above result is only **asymptotic** (holds in the limit)
- **Warning!** BCD may cycle indefinitely without converging, if the number of blocks is > 2 and the objective is nonconvex.

BCD – convergence

Two block BCD

$$\text{minimize}\, f(\boldsymbol{x}) = f(\boldsymbol{x}_1, \boldsymbol{x}_2) \quad \boldsymbol{x} \in \mathcal{X}_1 \times \mathcal{X}_2.$$

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Theorem (Grippo & Sciandrone (2000)). Let f be continuously differentiable, and the sets $\mathcal{X}_1$, $\mathcal{X}_2$ be closed and convex. Assume that the both BCD subproblems have solutions, and that the sequence $\{\boldsymbol{x}^k\}$ has limit points. Then, every limit point of $\{\boldsymbol{x}^k\}$ is stationary.

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- No need of **unique solutions** to subproblems
- BCD for 2 blocks is also called: **Alternating Minimization**

CD for convex problems

CD for smooth convex problems

$$\boxed{\min f(Ax) + \langle b,\, x\rangle \text{ subject to } x \geq 0}$$

- Function f is strictly convex and smooth
- Matrix $A \in \mathbb{R}^{m \times n}$ (possibly rank-deficient)

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- Apply CD to this problem
- With some more assumptions: it works!
- Even rate of convergence analysis (asymptotic)
- Here's the theorem

CD – convergence theorem

Assumptions:

1. Matrix A has no entirely zero column

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Theorem (Luo, Tseng (1992)). Let $\{x^k\}$ be a sequence of iterates generated by the CD method using the almost cyclic or the Gauss-Southwell rule for picking indices. Then, $\{x^k\}$ converges at least linearly to an element of $\mathcal{X}^*$.

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Proof is intricate; see Luo & Tseng's paper on bSpace.

CD – application

Projection onto convex sets

$$\boxed{\begin{array}{ll} \min & \frac{1}{2}\|x-y\|_2^2 \\ \text{s.t.} & x \in C_1 \cap C_2 \cap \cdots \cap C_m. \end{array}}$$

CD – application

Projection onto convex sets

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Solution 1: Rewrite using indicator functions

$$\min \quad \tfrac{1}{2}\|x-y\|_2^2 + \sum_{i=1}^{m} \delta_{C_i}(x).$$

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Solution 2: Take dual of the above formulation

Convex calculus time

$$\boxed{\min \quad \tfrac{1}{2}\|x - y\|_2^2 + f(x) + h(x)}$$

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$$L(x,z,w,\nu,\mu) := \tfrac{1}{2}\|x-y\|_2^2 + f(z) + h(w) + \nu^T(x-z) + \mu^T(x-w)$$

$$\begin{aligned}
g(\nu,\mu) &\quad:=\quad \inf_{x,z,w} L(x,z,\nu,\mu) \\
x-y+\nu+\mu=0 &\quad\Longrightarrow\quad x = y-\nu-\mu \\
g(\nu,\mu) &\quad=\quad -\tfrac{1}{2}\|\nu+\mu\|_2^2 + (\nu+\mu)^T y - f^*(\nu) - h^*(\mu)
\end{aligned}$$

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Dual as minimization problem

$$\boxed{\min \; k(\nu,\mu) := \tfrac{1}{2}\|\nu+\mu-y\|_2^2 + f^*(\nu) + h^*(\mu)}$$

Proximal-Dykstra method as CD

Apply CD to $k(\nu, \mu)$

$$\begin{aligned} \nu_{k+1} &= \operatorname{argmin}_{\nu}\ k(\nu, \mu_k) \\ \mu_{k+1} &= \operatorname{argmin}_{\mu}\ k(\nu_{k+1}, \mu) \end{aligned}$$

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- $0 \in \nu + \mu_k - y + \partial f^*(\nu)$

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- $0 \in \nu_{k+1} + \mu - y + \partial h^*(\mu)$
- $y - \mu_k \in \nu + \partial f^*(\nu) = (I + \partial f^*)(\nu)$
  $\implies \nu = \operatorname{prox}_{f^*}(y - \mu_k)$

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$$\boxed{\begin{aligned} \nu_{k+1} &\leftarrow y - \mu_k - \operatorname{prox}_f(y - \mu_k) \\ \mu_{k+1} &\leftarrow y - \nu_{k+1} - \operatorname{prox}_h(y - \nu_{k+1}) \end{aligned}}$$

Proximal-Dykstra as CD

- Simplify, and use Lagrangian stationarity to obtain primal

$$x = y - \nu - \mu \implies y - \mu = x + \nu$$

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- Thus, the CD iteration may be rewritten as

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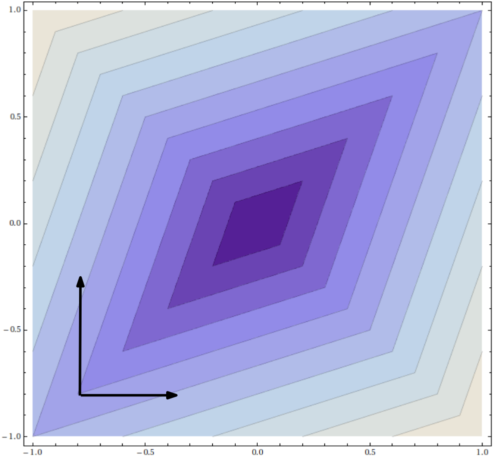
- This is the proximal-Dykstra method!

CD for nonsmooth convex problems

$$\min |x_1 - x_2| + \frac{1}{2}|x_1 + x_2|$$



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Remark: A related result for **nonconvex** problems with separable non-smoothness (under more assumptions), can be found in: "*Convergence of Block Coordinate Descent Method for Nondifferentiable Minimization*" by P. Tseng (2001).

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Choose $x_0 \in \mathbb{R}^n$. Let $M = \max_i L_i$; For $k \geq 0$

$$i_k = \mathrm{argmax}_{1 \leq i \leq n} |\nabla_i f(x_k)|$$
$$x_{k+1} = x_k - \tfrac{1}{M} \nabla_{i_k} f(x_k) e_{i_k}.$$

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Theorem Let $\{x^k\}$ be iterate sequence generated by above greedy CD method. Then,

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- With full gradient, might as well use ordinary gradient methods!
- Also, if $f \in C_L^1$, it can easily happen that $M \geq L$
- So above rate is in general, worse than gradient methods

Randomized CD

NEXT LECTURE:

- Randomized BCD (aka Stochastic BCD)
- Parallel BCD
- Dual decomposition, ADMM, etc.